Classical gravitational spin-spin interaction

W.B.Bonnor

School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road,
London. E1 4NS. UK

Abstract

I obtain an exact, axially symmetric, stationary solution of Einstein's equations for two massless spinning particles. The term representing the spin-spin interaction agrees with recently published approximate work. The spin-spin force appears to be proportional to the inverse fourth power of the coordinate distance between the particles.

1 Introduction

Papers on exact stationary solutions of Einstein's equations for pairs of massive spinning particles have been published by several authors [1-8]. These, though ingenious and important, particularly in the study of the equilibrium of black holes, contain no general formulae for the interactions between the particles. Recently I have studied this problem [10] by a method of approximation and obtained some general formulae which include interaction terms up to the second approximation in the masses and angular momenta of the particles.

There exists a class of exact solutions which offers important information on this subject in a surveyable form. This is the Papapetrou class [9], which is stationary and axially symmetric, and depends on one harmonic function. Using it one can construct a solution representing two particles bearing angular momentum but no mass. Because of the masslessness the solution is incomplete, of course, but nevertheless it gives an exact account of spin-spin gravitational interaction. This provides a check on the interaction obtained in [10]. The agreement turns out to be satisfactory, which boosts one's confidence in the approximation method of [10].

The plan of the paper is as follows. In section 2 I give the metric and the field equations, and in section 3 I describe Papapetrou's class of solutions and specialise it to the case of two spinning particles. The physical interpretation is discussed in section 4 where I give an estimate of the spin-spin force, and there is a concluding section 5.

2 Metric and field equations

We start from the metric for a stationary axially symmetric spacetime:

$$ds^2 = -f^{-1}e^{\nu}(dz^2 + dr^2) - ld\theta^2 - 2nd\theta dt + f dt^2, \tag{1}$$

where f, ν, l, n are functions of r and z. We use the vacuum field equations, sources being represented by singularities, and number the coordinates

$$x^1 = z, x^2 = r, x^3 = \theta, x^4 = t,$$

where

$$\infty > z > -\infty,\ r > 0,\ 2\pi \geq \theta \geq 0,\ \infty > t > -\infty,$$

$\theta = 0$ and $\theta = 2\pi$ being identified. Proceeding as in [10], we find that the field equations

$$R_{ik} = 0$$

can be put in the form

$$\begin{aligned}
R_{11} + R_{22} = \nu_{11} + \nu_{22} - f^{-1}\nabla^2 f + \frac{3}{2}f^{-2}(f_1^2 + f_2^2) & & \\
-\frac{1}{2}r^{-2}f^2(w_1^2 + w_2^2) &= 0, & (2)\\
R_{11} - R_{22} = r^{-1}\nu_2 + \frac{1}{2}f^{-2}(f_1^2 - f_2^2) + \frac{1}{2}r^{-2}f^2(w_2^2 - w_1^2) &= 0, & (3)\\
2R_{12} = -r^{-1}\nu_1 + f^{-2}f_1 f_2 - r^{-2}f^2 w_1 w_2 &= 0, & (4)\\
R_4^4 - R_3^3 - 2wR_4^3 = e^{-\nu}[-\nabla^2 f + f^{-1}(f_1^2 + f_2^2) - r^{-2}f^3(w_1^2 + w_2^2)] &= 0, & (5)\\
2R_4^3 = -r^{-2}f^2 e^{-\nu}[f\nabla^{*2}w + 2(f_1 w_1 + f_2 w_2)] &= 0, & (6)
\end{aligned}$$

where suffices 1 and 2 on the right denote $\partial/\partial z$ and $\partial/\partial r$, respectively, and

$$\begin{aligned}
w &= nf^{-1},\\
\nabla^2 X &= X_{11} + X_{22} + r^{-1}X_2,\\
\nabla^{*2} X &= X_{11} + X_{22} - r^{-1}X_2,
\end{aligned}$$

and l is given by

$$lf + n^2 = r^2. \tag{7}$$

3 The exact solution

Papapetrou's class of solutions of (2)-(7) is [1]

$$\begin{aligned}
f^{-1} &= \cosh\xi_1, & (8)\\
n &= rf\xi_2, & (9)\\
\nu_1 &= r\xi_{11}\xi_{12}, & (10)\\
\nu_2 &= \frac{1}{2}r[(\xi_{12})^2 - (\xi_{11})^2], & (11)\\
\nabla^2\xi &= 0, & (12)\\
l &= f^{-1}(r^2 - n^2), & (13)
\end{aligned}$$

[1] One may take $f^{-1} = \alpha\cosh\xi_1 + \beta\sinh\xi_1$, where α and β are constants. I put $\beta = 0$ to exclude unphysical mass dipoles, and $\alpha = 1$ to ensure $f = 1$ at infinity.

where ξ is a function of z and r, and a subscript 1 or 2 means differentiation as described above.

A solution for two particles on the z-axis at $z = \pm b$, with spins parallel or antiparallel, arises if we choose

$$\xi = -2\left(\frac{h_1}{R_1} + \frac{h_2}{R_2}\right), \tag{14}$$

where h_1 and h_2 are constants describing the angular momenta of the particles, and $R_1 =\mid [(z-b)^2 + r^2]^{1/2} \mid$, $R_2 =\mid [(z+b)^2 + r^2]^{1/2} \mid$. This gives

$$f^{-1} = \cosh 2\left(\frac{h_1(z-b)}{R_1^3} + \frac{h_2(z+b)}{R_2^3}\right), \tag{15}$$

$$n = 2fr^2\left(\frac{h_1}{R_1^3} + \frac{h_2}{R_2^3}\right), \tag{16}$$

$$\nu = \sum_{i=1}^{2}\frac{h_i^2 r^2(9r^2 - 8R_i^2)}{2R_i^8} - \frac{h_1 h_2[3(r^2+z^2-b^2)^3 + 2b^2r^2(9r^2+9z^2-b^2)]}{2b^4R_1^3R_2^3}$$
$$+C, \tag{17}$$

where C is an arbitrary constant. l can be obtained from (7),(15) and (16).

4 Physical interpretation

We see that n represents the spin term for two particles of angular momenta h_1, h_2 at points $z = \pm b$ on the z-axis. However, expanding f in inverse powers of R_1 and R_2 we have

$$f = 1 - 2\left(\frac{h_1(z-b)}{R_1^3} + \frac{h_2(z+b)}{R_2^3}\right)^2 + O(R^{-8}),$$

where $R^2 = z^2 + r^2$. This contains no term of order R^{-1}, i.e. no term representing mass. Thus we are modelling massless spinning particles; nevertheless, our solution will describe the spin-spin interaction. I shall comment further on this in the Conclusion.

The spin-spin interaction appears in the expression for ν. For Euclidean geometry on the axis we need

$$\lim_{r\to 0}\nu = 0, \tag{18}$$

and so from (17),

$$C - \frac{3h_1h_2(z^2-b^2)^3}{2b^4R_1^3R_2^3} = 0. \tag{19}$$

The arbitrary constant C can be chosen to satisfy this either for $\mid z \mid < b$ or for $\mid z \mid > b$, but not both. Let us choose

$$C = \frac{3h_1h_2}{2b^4}, \tag{20}$$

so that (19) is satisfied for $\mid z \mid > b$; then there is a conical singularity for $\mid z \mid < b$, i.e. between the particles. I interpret this in the usual way as a massless rod holding the particles in position, countering the spin-spin interaction.

The value (20) of C is the same as the spin-spin part of the corresponding constant in [10,eqn (28)], which adds to the credibility of the approximation method used there.

To quantify the spin-spin force I proceed as follows. Consider the metric near $r = 0$ for $\mid z \mid< b$. Rescaling the coordinates by introduction of ρ and Z by

$$\rho = re^C, Z = ze^C,$$

one obtains, neglecting terms of order ρ^2 except in $g_{\theta\theta}$,

$$ds^2 = -f_0^{-1}(dZ^2 + d\rho^2 + \rho^2 e^{-2C} d\theta^2) + f_0 dt^2, . \tag{21}$$

where $f_0 = \lim_{r\to 0} f, \mid z \mid< b$, so f_0 depends on z.

If f_0 were a constant (21) would be the metric for a (finite) cosmic string. Provided h_1 and h_2 are small, and the point on the axis under consideration is not too near the particles, f_0 is approximately equal to unity, so let us assume that the conical singularity in (21) does represent a cosmic string. Then its linear density is $\lambda = \frac{1}{4}(1 - e^{-C})$ [11]. The equation of state of the cosmic string is $p + \lambda = 0$, and it exerts no gravitational field, but it does exert a force [2] on the particles at its ends:

$$\begin{aligned} p &= -\frac{1}{4}(1 - e^{-C}) \sim -\frac{1}{4}C & (22) \\ &= -\frac{3h_1h_2}{8b^4}, & (23) \end{aligned}$$

which is a measure of the spin-spin force, repulsive if the spins are parallel.

This formula is no more than suggestive. Moreover, it cannot be given a rigorous meaning in terms of proper distance as the segment $\mid z \mid< b$ of the axis is singular. Note, however, that in the approximate calculations of [10], which take account of the ordinary gravitational attraction of the particles, the constant $\overset{(2)}{C}$, which corresponds to C here, is

$$\overset{(2)}{C} = \frac{3h_1h_2}{2b^4} - \frac{m_1m_2}{b^2},$$

so that (22) gives in this case

$$p = \frac{m_1m_2}{(2b)^2} - \frac{3h_1h_2}{8b^4},$$

which includes the expected inverse square law gravitational force. This suggests that the argument leading to (22) may be justified. Formula (22) is also compatible with the results of other authors [13] [14] [15] in the purely gravitational case.

5 Conclusion

Papapetrou's class of exact solutions includes one for two massless spinning particles in an axially symmetric configuration. This contains a conical singularity between the particles representing a strut balancing the spin-spin force. A tentative argument suggests that this force is given by (23). The calculations are consistent with those of [10], in which the interaction was treated by an approximation method.

In [10] it was shown that between massive spinning particles there is, in general, a *torsion singularity* in addition to the conical singularity. The torsion singularity depends

[2] In units of customary dimensions the force is $(-3Gh_1h_2)/(8c^2b^4)$.

on the masses of the particles as well as their angular momenta, and does not appear in the solution of this paper because in it the masses are zero.

The analogy between the gravitational spin-spin force and the force between two magnetic dipoles has been investigated by Wald [12]. In a subsequent paper I shall illustrate this analogy by deriving an exact solution of Einstein- Maxwell theory for two (massless) magnetic dipoles, and comparing it with the solution in this paper.

References

[1] Kramer D and Neugebauer G 1980 *Phys. Lett. A* **75** 259
[2] Dietz W and Hoenselaers c 1982 *Phys. Rev. Lett.* **48** 778
[3] Dietz W and Hoenselaers C 1985 *Ann. Phys. NY* **165** 319
[4] Hoensalaers C 1984 *Prog. Theor. Phys.* **72** 761
[5] Kihara M and Tomimatsu A 1982 *Prog. Theor. Phys.* **67** 349
[6] Dietz W 1984 *Solutions of Einstein's Equations* (Berlin: Springer) p85
[7] Manko V S, Ruiz E and Gómez 2000 *Class. Quantum Grav.* **17** 3881
[8] Manko V S and Ruiz E 2001 *Class. Quantum Grav.* **18** L11
[9] Papapetrou A 1953 *Annalen der Physik* **12** 309
[10]Bonnor W B 2001 *Class. Quantum Grav.* **18** 1381
[11]Hiscock W A 1985 *Phys. Rev.* D **31** 3288
[12]Wald R 1972 *Phys. Rev.* D**6** 406
[13]Robertson H P and Noonan T W 1968 *Relativity and Cosmology* (Philadelphia: W B Saunders) page 278
[14]Antoci S, Liebscher D-E and Mihich L 2001 *Class. Quantum Grav.* **18** 3463
[15]Katz A 1968 *J.Math. Phys.* **9** 983